CHANGES OF COORDINATES IN PLANE, CONICS

01. a. Rotation of $\pi/2$: The change is $\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$. Explicitly: $\begin{cases} x = -Y \\ y = X \end{cases}$ $\begin{cases} X = y \\ Y = -x \end{cases}$

b. Rotation of $\pi/2$ and new origin in $\{x = 2 \;;\; y = 0\}$

$$\begin{pmatrix} x-2 \\ y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \begin{cases} x = -Y + 2 \\ y = X \end{cases} \quad \begin{cases} X = y \\ Y = -x + 2 \end{cases}$$

c. Rotation of π and new origin in $\{x = 2 \;;\; y = 2\}$

$$\begin{pmatrix} x-2 \\ y-2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \begin{cases} x = -X + 2 \\ y = -Y + 2 \end{cases} \quad \begin{cases} X = -x + 2 \\ Y = -y + 2 \end{cases}$$

d. Rotation of $-\pi/6$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \begin{cases} x = (\sqrt{3}X + Y)/2 \\ y = (-X + \sqrt{3}Y)/2 \end{cases} \quad \begin{cases} X = (\sqrt{3}x - y)/2 \\ Y = (x + \sqrt{3}y)/2 \end{cases}$$

e. Rotation of $-\pi/6$ and new origin in $\{x = x_0 \;;\; y = 1\}$

Since X axis is $y = -\tan(\pi/6)x$, we easily find that $x_0 = -\sqrt{3}$

$$\begin{pmatrix} x-\sqrt{3} \\ y-1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad \begin{cases} x = \sqrt{3} + (\sqrt{3}X + Y)/2 \\ y = 1 + (-X + \sqrt{3}Y)/2 \end{cases}$$

Inverse is $\begin{cases} X = (\sqrt{3}(x - \sqrt{3}) - (y-1))/2 \\ Y = ((x - x_0) + \sqrt{3}(y-1))/2 \end{cases}$ or more simply $\begin{cases} X = (\sqrt{3}x - y + 4)/2 \\ Y = (x + \sqrt{3}y)/2 \end{cases}$

f. Note that the *absolute value* of X is the distance form Y axis and so for Y. So the change should be $\begin{cases} X = \pm(x + 2y - 1)/\sqrt{5} \\ Y = \pm(2x - y - 5)/\sqrt{5} \end{cases}$.

The signs $\pm$ must be choosen looking at the directions of X and Y axes. So substitute in the equations the values $\{x = 0 \;;\; y = 0\}$. We get $\{X = \pm(-1)/\sqrt{5} \;;\; Y = \pm(-5)/\sqrt{5}\}$. From the picture we see that the point $\{x = 0; y = 0\}$ has positive coordinate X and negative coordinate Y, then the signs must be: "$-$" for first equation and "$+$" for the second, that is

$$\begin{cases} X = (-x - 2y + 1)/\sqrt{5} \\ Y = (2x - y - 5)/\sqrt{5} \end{cases} \longrightarrow \begin{cases} X - 1/\sqrt{5} = (-x - 2y)/\sqrt{5} \\ Y + 5/\sqrt{5} = (2x - y)/\sqrt{5} \end{cases}$$

This is the inverse change. The direct one is obtained by transposing the matrix:

$$\begin{cases} x = (-(X - 1/\sqrt{5}) + 2(Y + 5/\sqrt{5}))/\sqrt{5} \\ y = (-2(X - 1/\sqrt{5}) - (Y + 5/\sqrt{5}))/\sqrt{5} \end{cases} \longrightarrow \begin{cases} x = (-X + 2Y)/\sqrt{5} + 11/5 \\ y = (-2X - Y)/\sqrt{5} - 3/5 \end{cases}$$

02. The two lines are orthogonal, this means that they can be chosen as axes of a cartesian system. So the change (inverse formula) must be of the following kind;

$\begin{cases} X = \pm(11x + 12y - 10)/\sqrt{265} \\ Y = \pm(12x - 11y + 8)/\sqrt{265} \end{cases}$ since the matrix must be orthogonal. We don't know the directions of new axes. But the orthogonal matrix should have determinant 1. This can be achieved for these choices of signs: "$+\ -$" or "$-\ +$".

Direct change is: $\begin{cases} x = \pm(11X - 12Y)/\sqrt{265} + 14/265 \\ y = \pm(-12X - 11Y)/\sqrt{265} + 208/265 \end{cases}$ Signs should again be "$+\ -$" or "$-\ +$".

03. a. It's a rotation of $5\pi/4$: The change is:

$$\begin{cases} x = -(\sqrt{2}/2)X + (\sqrt{2}/2)Y \\ y = -(\sqrt{2}/2)X - (\sqrt{2}/2)Y \end{cases} \quad \begin{cases} X = -(\sqrt{2}/2)x - (\sqrt{2}/2)y \\ Y = (\sqrt{2}/2)x - (\sqrt{2}/2)y \end{cases}$$

The equation of the parabola, w.r. to system OXY, is $Y = aX^2$. To find a, we impose the condition of passing through $\{x = 0 \;;\; y = -1\}$ whose coordinates w.r. to OXY are $\{X = \sqrt{2}/2 \;;\; Y = \sqrt{2}/2\}$.

We find $a = \sqrt{2}$. The equation of the parabola w.r. to Oxy is found substituting in X and Y: $(\sqrt{2}/2)x - (\sqrt{2}/2)y = \sqrt{2}(-(\sqrt{2}/2)x - (\sqrt{2}/2)y)^2$ or $x^2 + 2xy + y^2 - x + y = 0$.

b. The inverse change is: $\begin{cases} X = (x+y)/\sqrt{2} \\ Y = (-x+y-2)/\sqrt{2} \end{cases}$ The semiaxes of the ellipse are 3 and $\sqrt{2}$, so the equation w.r. to $O'XY$ is: $\dfrac{X^2}{9} + \dfrac{Y^2}{2} = 1$. By substituting we get the equation w.r. to Oxy: $\dfrac{(x+y)^2}{18} + \dfrac{(-x+y-2)^2}{4} = 1$ or $11x^2 - 14xy + 11y^2 + 36x - 36y = 0$.

c. The inverse change of coordinates is

$$\begin{cases} X = (\sqrt{2}/2)(x-2) - (\sqrt{2}/2)(y-1) \\ Y = (\sqrt{2}/2)(x-2) + (\sqrt{2}/2)(y-1) \end{cases} \text{ or } \begin{cases} X = (x-y-1)/\sqrt{2} \\ Y = (x+y-3)/\sqrt{2} \end{cases}$$

The major semiaxis is $\sqrt{2}$ (the distance between points $\{x=3\ ;\ y=0\}$ and $\{x=2\ ;\ y=1\}$). So in the system $O'XY$ the equation is $\dfrac{X^2}{2} + \dfrac{Y^2}{b^2} = 1$.

The ellipse passes through the point $\{x=2\ ;\ y=0\}$, whose coordinates w.r. to $O'XY$ are $\{X = \sqrt{2}/2\ ;\ Y = -\sqrt{2}/2\}$. Substituting in the equation we get: $\dfrac{1/2}{2} + \dfrac{1/2}{b^2} = 1$ and so $b^2 = 2/3$. Therefore the conic is $\dfrac{X^2}{2} + \dfrac{3Y^2}{2} = 1$.

The equation in the system Oxy is gotten by substituting:

$$\frac{\left((x-y-1)/\sqrt{2}\right)^2}{2} + \frac{3\left((x+y-3)/\sqrt{2}\right)^2}{2} = 1 \text{ or } x^2 + xy + y^2 - 5x - 4y + 6 = 0.$$

CHANGES OF COORDINATES IN SPACE

10. Direction vectors of X, Y, Z axes are respectively:
$(\cos(\alpha)\ ,\ \sin(\alpha)\ ,\ 0)\ ,\ (-\sin(\alpha)\ ,\ \cos(\alpha)\ ,\ 0)\ ,\ (0,0,1)$, so we can write the matrix which is orthogonal and has determinant 1

$$P = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ The formulas are: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P^T \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

11. Direction vectors of X, Z axes are respectively:
$\vec{v}_X(\cos(\pi/3)\ ,\ \sin(\pi/3)\ ,\ 0)\ ,\ \vec{v}_Z(-\sin(\pi/3)\ ,\ \cos(\pi/3)\ ,\ 0)$.
The direction vector $\vec{v}_Y$ of Y axis must be $\vec{v}_Z \wedge \vec{v}_X$ (and not $\vec{v}_X \wedge \vec{v}_Z$!), if we want right-handed coordinates) and so $\vec{v}_Y = (0,0,-1)$
Therefore the matrix is

$$P = \begin{pmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \\ 0 & -1 & 0 \end{pmatrix}. \text{ Formulas are: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P^T \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

12. First rotation around z axis is given by the formula

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Second rotation around X axis is given by the formula

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \\ Z'' \end{pmatrix}$$

Passage from (x,y,z) to (X'', Y'', Z'') is found substituting

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \\ Z'' \end{pmatrix} \text{ whence:}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha)\cos(\beta) & -\sin(\alpha)\cos(\beta) \\ \sin(\alpha) & \cos(\alpha)\cos(\beta) & -\cos(\alpha)\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \\ Z'' \end{pmatrix}$$

13. Direction vector of z axis is $\vec{v}_Z = (P - O) = (2, 1, 1)$. Since X axis lies on $[xy]$ plane, the third coordinate of its direction vector should be 0, that is $\vec{v}_X = (a, b, 0)$. The vector should be orthogonal to $\vec{v}_Z$, that is $(2, 1, 1) \cdot (a, b, 0) = 0$. We have $2a + b = 0$, for instance we can put $\vec{v}_X = (1, -2, 0)$ (we do not choose $(-1, 2, 0)$ since, as we can see from the picture, the first coordinate of X axis is positive).

    The direction vector $\vec{v}_Y$ of Y axis must be $\vec{v}_Z \wedge \vec{v}_X$ (and not $\vec{v}_X \wedge \vec{v}_Z$!), therefore $\vec{v}_Y = (2, 1, -5)$. Now normalize the three vectors to write the orthogonal matrix and the change of coordinates:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{30} & 2/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{30} & 1/\sqrt{6} \\ 0 & -5/\sqrt{30} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

14. The X axis can be written in parametric representation as $\{x = 2t; y = t; z = 2t\}$. Its direction vector is $\vec{v}_X = (2, 1, 2)$. Just for Y, a parametric representation can be $\{x = k - t; y = 1; z = t\}$ and is direction vector is $\vec{v}_Y = (-1, 0, 1)$.
    The two vectors are orthogonal, but the two axes should meet in some point, so the near linear system should have a solution. It can be easily seen that this is true only when $k = 4$.
    $$\begin{cases} 2t = k - u \\ t = 1 \\ 2t = u \end{cases}$$
    In this case the two lines meet in the point $O'(2, 1, 2)$. This point will be the new origin. The direction vector of Z axis shall be $\vec{v}_X \wedge \vec{v}_Y = (1, 4, 1)$. The direction vector of X axis should be normalized, but there are two choices: $(2/3, 1/3, 1/3)$ or the opposite vector and so is for Y axis. The direction vector of Z axis should be oriented following the four possible choices. There are four possible changes of coordinates.

$$\begin{pmatrix} x-2 \\ y-1 \\ z-2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ 1/2 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad \begin{pmatrix} x-2 \\ y-1 \\ z-2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/2 & 0 & -4/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & -1/\sqrt{18} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} x-2 \\ y-1 \\ z-2 \end{pmatrix} = \begin{pmatrix} -2/3 & -1/\sqrt{2} & -1/\sqrt{18} \\ -1/2 & 0 & -4/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad \begin{pmatrix} x-2 \\ y-1 \\ z-2 \end{pmatrix} = \begin{pmatrix} -2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ -1/2 & 0 & 4/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

15. Normal vectors of the two planes are respectively $\vec{n}_1(1, 2, -1)$ and $\vec{n}_2(1, 0, 1)$. These (ore their opposite) shall be the directional vectors of Z axis and of X axis. The origin O' must lie on the line intersection of the two planes (the Y axis). The directional vectore of this line is $\vec{v}_Y = (1, -1, -1)$ (multiple of $\vec{n}_1 \wedge \vec{n}_2$). The point $(1, 0, 0)$ lies on both planes. So the line is $\{x = 1 - t\,;\ y = t\,;\ z = t\}$. Each point of this line can be can be chosen as new origin O'. The plane $[XZ]$ is orthogonal to Y axis and so its equation is $(x - 1 - t) - (y - t) - (z - t) = 0$. The following are the possible changes of coordinates (for any choice of t):

    Two of them are:
    $$\begin{pmatrix} x-1+t \\ y-t \\ z-t \end{pmatrix} = \begin{pmatrix} \pm 1/\sqrt{2} & \pm 1/\sqrt{3} & \pm 1/\sqrt{6} \\ 0 & \mp 1/\sqrt{3} & \pm 2/\sqrt{6} \\ \pm 1/\sqrt{2} & \mp 1/\sqrt{3} & \mp 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$
    All the other are obtained reversing "±" e "∓ only in the last two columns

Curves and surfaces

20. a. We can write $t = x$ and $t^2 = (z - 1)/2$. By substitution in $y = t - t^2$ we get the equation $y = x - (z - 1)/2$ or $2x - 2y - z + 1 = 0$. This is the equation of a plane that is satisfied by every point of $\mathcal{L}$. So $\mathcal{L}$ is a plane curve contained in that plane.

    b. The orthogonal projection of $\mathcal{L}$ onto $z = 0$ is simply $\mathcal{L}_0$.
    $$\mathcal{L}_0 \begin{cases} x = t \\ y = t - t^2 \\ z = 0 \end{cases}$$
    To get the orthogonal projection onto $x = y$, we must write the cylinder $\mathcal{C}_1$ containing $\mathcal{L}$ with generatrices orthogonal to the plane $x = y$. The cylinder is made out of all the lines passing through the point $(t\ ,\ t - t^2\ ,\ 2t^2)$ and parallel to the vector $(1, -1, 0)$, hence its parametric repesentation.
    $$\mathcal{C}_1 \begin{cases} x = t + u \\ y = t - t^2 - u \\ z = 2t^2 + 1 \end{cases}$$
    Now intersect the cylinder and $x = y$. We get $t + u = t - t^2 - u$ i.e. $u = -t^2/2$. Substituting in the cylinder we get the projection $\mathcal{L}_1$.
    $$\mathcal{L}_1 \begin{cases} x = t - t^2/2 \\ y = t - t^2/2 \\ z = 2t^2 + 1 \end{cases}$$

    c. Eliminate t. We get $\begin{cases} y = x - x^2 \\ z = 2x^2 + 1 \end{cases}$ or, using the fact the the curve is plane $\begin{cases} y = x - x^2 \\ 2x - 2y - z + 1 = 0 \end{cases}$

21. a. Suppose that there exists a plane $\alpha : ax+by+cz+d=0$ containing all the points of $\mathcal{L}$. We would have $a(t+1)+b(t^2)+c(t^3)+d=0$ for each t, so the polynomial $ct^3+bt^2+at+(a+d)$ should be *identically* null, but this is true only if $a=b=c=d=0$. This means that there is no plane containing $\mathcal{L}$.

b. The procedure is the same as in the preceding exercise. We get $\mathcal{L}_0$ and $\mathcal{L}_1$ $\mathcal{L}_0\begin{cases} x &= t+1\\ y &= t^2 \\ z &= 0\end{cases}$ $\mathcal{L}_1\begin{cases} x &= (t^2+t+1)/2\\ y &= (t^2+t+1)/2 \\ z &= t^3\end{cases}$

c. Just eliminate t: $\begin{cases} y &= (x-1)^2\\ z &= (x-1)^3\end{cases}$

d. A parametric representation of the three cylinders is easily obtained writing all the lines passing through the point $(t+1\ ,\ t^2\ ,\ t^3)$ and parallel respectively to the vectors $(0,0,1),(1,0,0),(2,1,2)$:

$$\mathcal{C}_1\begin{cases} x &= t+1\\ y &= t^2\\ z &= t^3+u\end{cases}\qquad \mathcal{C}_2\begin{cases} x &= t+1+u\\ y &= t^2\\ z &= t^3\end{cases}\qquad \mathcal{C}_3\begin{cases} x &= t+1+2u\\ y &= t^2+u\\ z &= t^3+2u\end{cases}$$

The cartesian representations of $\mathcal{C}_1$ and $\mathcal{C}_2$ are straightforward and are $y=(x-1)^2\ ;\ z^2=y^3$. The cartesian representations of $\mathcal{C}_3$ requires some rather complicated substitution in order to eliminate u and t. First eliminate u and get two equations

$\begin{matrix}E_1\\E_2\end{matrix}\begin{cases} 2y-2t^2=x-t-1\\ z-t^3=x-t-1\end{cases}$ Then eliminate t with the following operations: $\begin{matrix}E_1\to E_1\\ E_2\to 2E_2-tE_1\end{matrix}$

$$\begin{cases} 2y-2t^2=x-t-1\\ 2z-2ty=2x-tx+t^2-t-2\end{cases}\quad \begin{matrix}E_1\to E_1\\ E_2\to 2E_2-E_1\end{matrix}\quad \begin{cases} 2y-2t^2=x-t-1\\ 4z-2y-3x+3=t(-2x+4y-1)^2\end{cases}$$

Now solve E_2 for t, substitute in E_1 and get the cartesian equation of $\mathcal{C}_3$:

$$2y(2x-4y+1)^2-2(3x+2y-4z-3)^2=$$
$$=x(2x-4y+1)^2-(2x-4y+1)(3x+2y-4z-3)-(2x-4y+1)^2$$

But to solve E_2, we had to divide by the polynomial $2x-4y+1$ and so this cylinder contains the line $\{2x-4y+1;3x+2y-4z-3\}$ which was not in the parametric $\mathcal{C}_3$.

e. Parametric representation of the cones are easily obtained writing all the lines passing through the point $(t+1\ ,\ t^2\ ,\ t^3)$ and respectively the point $(0,0,0)$ and the point $(1,-1,2)$

The cones are: $\begin{cases} x &= u(t+1)\\ y &= ut^2\\ z &= ut^3\end{cases}$ $\begin{cases} x &= 1+u(t+1)\\ y &= -1+ut^2\\ z &= 2+ut^3\end{cases}$

One can easily eliminate t e u from the first cone and get $z^2x=zy^2+y^3$ (but it contains the z axis: $\{y=0\ ;\ x=0\}$ that was not a line of the parametric cone). The second cone is simply: $(z-2)^2(x-1)=(z-2)(y+1)^2+(y+1)^3$ (there is again one more line: $\{y+1=0\ ;\ x=1\}$).

22. a. The first three cylinders are easily found eliminating respectively x,y,z from the system: $(z+2y)^2=z^2(z+y)\qquad x^2+(x^2-z)-xz=0\qquad x^2+y-x(x^2-y)$.

The fourth cylinder is formed by the lines passing through the point (α,β,γ) and parallel to ℓ which are (1)$\begin{cases} x &= \alpha+t\\ y &= \beta+2t\\ z &= \gamma\end{cases}$ But (α,β,γ) must lie on L, that is (2)$\begin{cases} \alpha^2+\beta-\alpha\gamma &= 0\\ \alpha^2-\beta &= \gamma\end{cases}$

Eliminate α,β,γ,t from (1) using (2). We find:

$\begin{cases} \alpha^2 &= \beta+z\\ 2\beta+z-\alpha z &= 0\\ 2x-y &= 2\alpha-\beta\end{cases}$ $\begin{cases} \alpha^2 &= 2\alpha-2x+y+z\\ \alpha(4-z) &= 4x-2y-z\\ \beta &= 2\alpha-2x+y\end{cases}$ After easy passages we get: $(4x-2y-z)^2=2(4x-2y-z)(4-z)+(-2x+y+z)(4-z)^2$.

b. By instance $\begin{cases}(z+2y)^2=z^2(z+y)\\ x^2+(x^2-z)-xz=0\end{cases}$ (two of the cylinders found in [a.])

30. a. We must write all the circles passing through the point (t,t,t) and having z axis as axis. They are $\{z=t\ ;\ x^2+y^2=2t^2\}$. The surface is easily found eliminating t and is $x^2-y^2-2z^2=0$. It is a quadric cone.

b. We must write all the circles passing through the point $(t, 1, t)$ and having z axis as axis. They are $\{z = t\ ;\ x^2 + y^2 = 1 + t^2\}$. The surface is $x^2 + y^2 - z^2 = 1$. It is an hyperboloid of one sheet.

c. The circles passing through the point $(2, t+1, t)$ and having z axis as axis have the representation $\{x = 2\ ;\ y^2 + z^2 = 2t^2 + 2t + 1\}$. The resulting surface is the plane $z = 2$, but since $2t^2 + 2t + 1 \geq 1/2$ for every t, in the surface the open disc with center $(0, 0, 2)$ and radius $\sqrt{2}/2$ is missing.

d. The circles passing through the point $(1, 2t^2, t)$ and having z axis as axis have the representation $\{z = t^2/2\ ;\ x^2 + y^2 = t^2\}$. The surface is $x^2 + y^2 = 2z$. It is an elliptic paraboloid.

e. The circles passing through the point $(t, 2t^2, t)$ and having z axis as axis have the representation $\{z = \beta\ ;\ x^2 + y^2 = 1 + 4t^4\}$. The surface is $x^2 + y^2 = 1 + 4z^4$. It is a quartic surface and it is hard to describe it in detail.

f. The circles passing through the point $(0, \alpha, \beta)$ (with $(\alpha - 2)^2 + \beta^2 = 1$ [1]) and having z axis as axis are $\{z = \beta\ ;\ x^2 + y^2 = \alpha^2\}$. Substitute α e β in the condition [1]. We get $x^2 + y^2 + z^2 - 4\alpha + 3 = 0$ or $4\alpha = x^2 + y^2 + z^2 + 3$. Hence, taking squares and using again the second equation of the circles, we get $16(x^2 + y^2) = (x^2 + y^2 + z^2 + 3)^2$. This is the surface. It is a "torus" (a surface shaped as a doughnut).

31. a. The equation can be written as $(x + y)(x - y) = xz$ or $\dfrac{x + y}{z} = \dfrac{x}{x - y} = a$. Hence we get the lines: $\begin{cases} x + y = az \\ x = a(x - y) \end{cases}$ $(a \in \mathbb{R})$. But since we divided by z and $x - y$, the line $\{z = 0\ ;\ x = y\}$, which lies on the surface, is missing in the family.
We can also write $\dfrac{x + y}{x} = \dfrac{z}{x - y} = b$ and get the lines $\begin{cases} x + y = bx \\ z = b(x - y) \end{cases}$ $(b \in \mathbb{R})$ (and now the line $\{x = 0\ ;\ x = y\}$ is missing).
The surface is doubly ruled (it is an hyperboloid of two sheets).

b. The equation can be written as $x \cdot y = z \cdot 1$. Like in the previous exercise we get two families of lines: $\begin{cases} x = az \\ ay = 1 \end{cases}$ $\begin{cases} x = b \\ by = z \end{cases}$ $(a, b \in \mathbb{R})$. Only the line $\{x = 0\ ;\ z = 0\}$ is missing. The surface is doubly ruled (it is an hyperbolic paraboloid).

c. Let $a = x$ and get $a^3 y = az + z$. The lines are $\{a^3 y = az + z\ ;\ x = a\}$

d. Let $a = z$ and get immediately the lines $\{x + 3y = a^3\ ;\ z = a\}$ (it is a cylinder).

e. Let $a = x - z$. The lines are: $\{x - z = a\ ;\ a^2 x = y + z\}$.

f. The equation can be written as $x \cdot x = y \cdot z$. Like in exercise [a.] the lines are: $\begin{cases} x = ay \\ ax = z \end{cases}$ $\begin{cases} x = bz \\ bx = y \end{cases}$ $(a, b \in \mathbb{R})$. But the surface is not doubly ruled. In fact, if we set $a = 1/b$ we get from the first family of lines all the lines of the second family, but there are two exceptions: in the first family the line $\{y = 0\ ;\ x = 0\}$ is missing, in the second family the line $\{z = 0\ ;\ x = 0\}$ is missing. The surface is a cone and so it is only single ruled.

g. Let $a^2 = z$. We have $x = ay$ or $x = -ay$. Therefore the lines are $\{z = a^2\ ;\ x = ay\}$ and $\{z = a^2\ ;\ x = -ay\}$. But the surface is not dubly ruled since in every point of the surface one can find only a line of the surface which belongs to the first family or to second family.

32. Just intersect the line ℓ with the cone. We get the points $P_1 = (2 - \sqrt{2}, -2 + 2\sqrt{2}, -1 + \sqrt{2})$ and $P_2 = (2 + \sqrt{2}, -2 - 2\sqrt{2}, -1 - \sqrt{2})$. The lines through $(0, 0, 0)$ (vertex of the cone) and resp. through P_1 and P_2 are the required lines. The lines lying on the cone are $\{x = ay\ ;\ ax = z\}$ (cf. 31.f.). They have parametric representation $\{x = at\ ;\ y = t\ ;\ z = a^2 t\}$ and so they are parallel to the vector $(a, 1, a^2)$. The line ℓ is parallel to the vector $(-1, 2, 1)$. So we must have $(a, 1, a^2) \cdot (-1, 2, 1) = 0$, that is $-a + 2 + a^2 = 0$. But this equation has no real solutions, so there are no such lines. We remark that in the family the line $\{y = 0\ ;\ x = 0\}$ was missing, but this line too is not orthogonal to ℓ.